# Part II

# Remarks on Drifts,
# Perpendicular & Anomalous Diffusion

# Outline

- Why going beyond ‘standard’ theory?
- Drifts by Turbulence?
- Perpendicular diffusion and an overlooked rotation?
- Anomalous diffusion in the heliosphere?
- Résumé

# Why Going Beyond Standard Theory?

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Because real physical systems are...

- ... **not homogeneous/not uniform:**
  $\longrightarrow$ Talk by Reinhard Schlickeiser: inhomogeneous magnetic field
  $\longrightarrow$ Talk by Stefan Artmann: inhomogeneous magnetic field
  $\longrightarrow$ Talk by Kobus le Roux: spiral magnetic field

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... and because it's fun to explore something new ☺.

# Drifts by Turbulence?

# Structure of the diffusion tensor

- homogeneous magnetic field, axisymmetric turbulence, zero helicity

$$\overleftrightarrow{\kappa}(\vec{r}, p, t) = \begin{pmatrix} \kappa_\perp & 0 & 0 \\ 0 & \kappa_\perp & 0 \\ 0 & 0 & \kappa_{||} \end{pmatrix} \quad ; \quad \kappa_{||} = \frac{v^2}{8} \int_{-1}^{1} \frac{(1-\mu)^2}{D_{\mu\mu}(\mu)}\, d\mu \quad \ldots$$

$\Longrightarrow$
- isotropic perpendicular diffusion
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- inhomogeneous m. f. and/or non-axisymmetric turb., non-zero hel.

$$\overleftrightarrow{\kappa}(\vec{r}, p, t) = \begin{pmatrix} \kappa_{\perp 1} & \kappa_{xy} & 0 \\ \kappa_{yx} & \kappa_{\perp 2} & 0 \\ 0 & 0 & \kappa_{||} \end{pmatrix} \quad ; \quad \kappa_{||} = \frac{v^2}{8} \int_{-1}^{1} \frac{(1-\mu)^2}{D_{\mu\mu}(\mu)} \, d\mu \quad \ldots$$

$\Longrightarrow$
- anisotropic perpendicular diffusion
- drifts

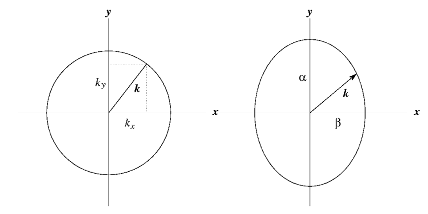
# Consequences of non-axisymmetric turbulence

Study of off-diagonal elements: **turbulence-induced drifts**

$$\kappa_{lm} = \lim_{t\to\infty} \frac{\langle \Delta x_l \Delta x_m \rangle}{2t} \;;\; \frac{d^2 \langle \Delta x_l \Delta x_m \rangle}{dz^2} = \frac{2}{B_0^2} \int P_{lm}(\vec{k}) \langle \exp\{i\vec{k}\cdot\Delta\vec{x}\}\rangle d^3k$$

$$P_{lm}(\vec{k}) = \langle \delta B_l(\vec{k}) \delta B_m^*(\vec{k}) \rangle = A(k_x, k_y)\,(k_\perp^2 \delta_{lm} - k_l k_m)$$

Lines $A(k_x, k_y) = const$:




axisymm. non-axisymm.

$\Rightarrow$

$$\kappa_{xy} = \kappa_{yx} = \frac{(\xi^2 - 1)\tan^2\Phi}{1 + \xi^2 \tan^2\Phi}\,\kappa_{\perp 1}$$

$$\frac{\kappa_{\perp 2}}{\kappa_{\perp 1}} = \frac{\tan^2\Phi + \xi^2}{1 + \xi^2 \tan^2\Phi} \;;\; \xi = \frac{\alpha}{\beta}$$

*Ruffolo et al. [2006,2008], Stawicki [2005]*
*Weinhorst et al. [2008]*

# Non-axisymmetric turbulence in the solar wind

## Large-Amplitude Alfvén Waves in the Interplanetary Medium, 2

J. W. Belcher and Leverett Davis, Jr.

*Physics Department, California Institute of Technology, Pasadena 91109*

An extensive study of the dynamic nonshock properties of the microscale fluctuations (scale lengths of 0.01 AU and less) in the interplanetary medium was made by using plasma and magnetic field data from Mariner 5 (Venus 1967). The observational results of the study are: (1) Large-amplitude, nonsinusoidal Alfvén waves propagating outward from the sun with a broad wavelength range from $10^3$ to $5 \times 10^6$ km dominate the microscale structure at least 50% of the time; the waves frequently have energy densities comparable both to the unperturbed magnetic field energy density and to the thermal energy density. (2) The purest examples of these outwardly propagating Alfvén waves occur in high-velocity solar wind streams and on their trailing edges (where the velocity decreases slowly with time). In low-velocity regions Alfvén waves are also outwardly propagating but usually have smaller amplitudes than in the fast streams and tend to be less pure in the sense that they are more strongly intermixed with structures of a non-Alfvénic and possibly static nature. (3) The largest amplitude Alfvénic fluctuations are found in the compression regions at the leading edges of high-velocity streams where the velocity increases rapidly with time; these regions may contain significant amounts of inwardly propagating or non-Alfvénic wave modes. (4) Power spectra of the interplanetary magnetic field in the frequency range from 1/(107 min) to 1/(25.2 sec) have frequency dependencies of $f^{-1.5}$ to $f^{-2.2}$; the spectra with slower falloffs tend to be associated with higher temperature regions. (5) The microscale magnetic field fluctuations have on the average a 5:4:1 power anisotropy in an orthogonal coodinate system whose axes are $(\mathbf{e}_B \times \mathbf{e}_R, \mathbf{e}_B \times (\mathbf{e}_B \times \mathbf{e}_R), \mathbf{e}_B)$, where $\mathbf{e}_B$ is a unit vector in the average field direction and $\mathbf{e}_R$ is a unit vector radially away from the sun; this anisotropy tends to be strongest (6:3:1) in the compression regions at the leading edges of high-velocity streams. (6) Presumably magneto-

# Perpendicular Diffusion and an Overlooked Rotation?

# The Diffusion Tensor & An Overlooked Rotation?

Study of large-scale transport requires transformation of the diffusion tensor from a (field-aligned) local to a global system:

- Which local system is the most 'natural' one?

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- The correct choice of the 'perpendicular' directions is important for the case of anisotropic perpendicular diffusion (e.g. for Jovian electrons).

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Study of large-scale transport requires transformation of the diffusion tensor from a (field-aligned) local to a global system:

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- One principal direction aligned with magnetic field.
- The correct choice of the 'perpendicular' directions is important for the case of anisotropic perpendicular diffusion (e.g. for Jovian electrons).
- The most natural directions are defined with the Frenet-Serret trihedron, i.e. by the curvature and torsion of a magnetic field:

$$\vec{t} = \vec{B}/B \quad ; \quad \vec{n} = (\vec{t}\cdot\nabla)\,\vec{t}/k \quad ; \quad \vec{b} = \vec{t}\times\vec{n}$$

This has not been considered for (heliospheric) CR transport, so far...

# The Diffusion Tensor & An Overlooked Rotation?

The **general transformation** reads:

$$\overleftrightarrow{\kappa}_{global} = A^T \; \overleftrightarrow{\kappa}_{local} \; A$$

with $A = \begin{pmatrix} n_1 & b_1 & t_1 \\ n_2 & b_2 & t_2 \\ n_3 & b_3 & t_3 \end{pmatrix}$ and $\overleftrightarrow{\kappa}_{local} = \begin{pmatrix} \kappa_{\perp 1} & 0 & 0 \\ 0 & \kappa_{\perp 2} & 0 \\ 0 & 0 & \kappa_{||} \end{pmatrix}$

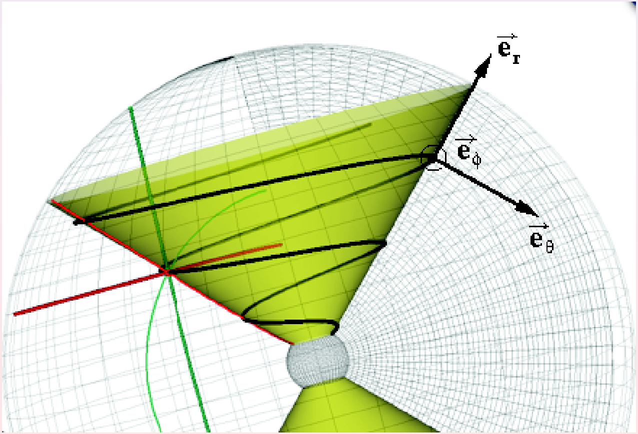
resulting in

$$\begin{aligned}
\kappa_{11} &= \kappa_{\perp 1}\, n_1^2 + \kappa_{\perp 2}\, b_1^2 + \kappa_{||}\, t_1^2 \\
\kappa_{12} &= \kappa_{\perp 1}\, n_1\, n_2 + \kappa_{\perp 2}\, b_1\, b_2 + \kappa_{||}\, t_1\, t_2 \\
\kappa_{13} &= \kappa_{\perp 1}\, n_1\, n_3 + \kappa_{\perp 2}\, b_1\, b_3 + \kappa_{||}\, t_1\, t_3 \\
\kappa_{22} &= \kappa_{\perp 1}\, n_2^2 + \kappa_{\perp 2}\, b_2^2 + \kappa_{||}\, t_2^2 \\
\kappa_{23} &= \kappa_{\perp 1}\, n_2\, n_3 + \kappa_{\perp 2}\, b_2\, b_3 + \kappa_{||}\, t_2\, t_3 \\
\kappa_{33} &= \kappa_{\perp 1}\, n_3^2 + \kappa_{\perp 2}\, b_3^2 + \kappa_{||}\, t_3^2
\end{aligned}$$

# The Diffusion Tensor & An Overlooked Rotation?

Example: **Parker Field**

'Classical' Choice: $\kappa_{\perp 2}$ always along $\vec{e}_\theta$




$$\begin{pmatrix} \kappa_{\perp r} & 0 & 0 \\ 0 & \kappa_{\perp\theta} & 0 \\ 0 & 0 & \kappa_{||} \end{pmatrix}$$

$\downarrow$

*transformation*

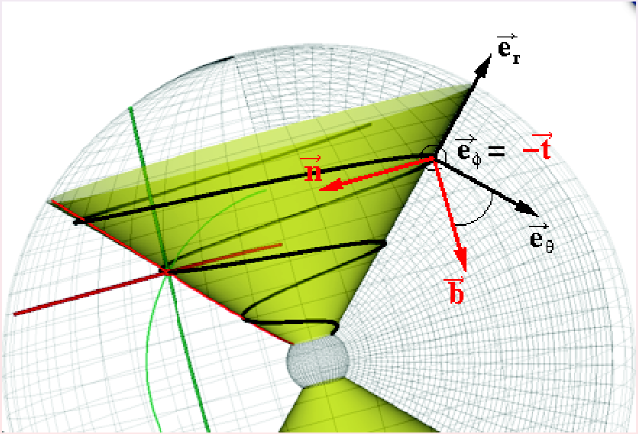
$\downarrow$

$$\begin{pmatrix} \kappa_{\perp r} & 0 & \kappa_{r\phi} \\ 0 & \kappa_{\perp\theta} & 0 \\ \kappa_{r\phi} & 0 & \kappa_{||} \end{pmatrix}$$

# The Diffusion Tensor & An Overlooked Rotation?

Example: **Parker Field**

'Natural' Choice: $\kappa_{\perp 2}$ along $\vec{b} = \vec{t} \times \vec{n}$




$$\begin{pmatrix} \kappa_{\perp 1} & 0 & 0 \\ 0 & \kappa_{\perp 2} & 0 \\ 0 & 0 & \kappa_{\|} \end{pmatrix}$$

$\downarrow$

*transformation*

$\downarrow$

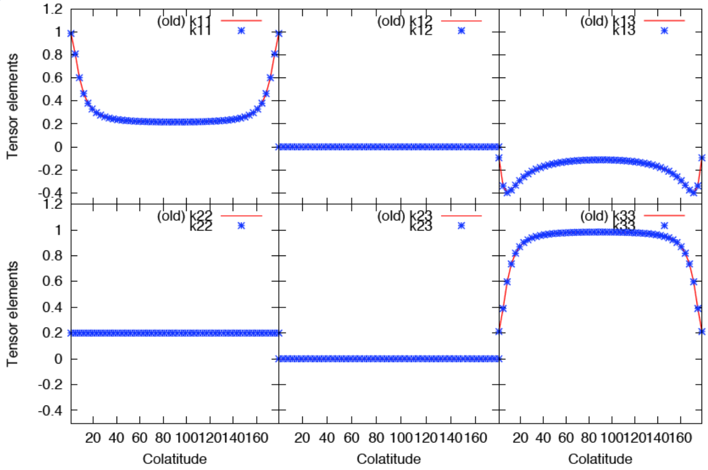
$$\begin{pmatrix} \kappa_{\perp 1} & \kappa_{r\theta} & \kappa_{r\phi} \\ \kappa_{r\theta} & \kappa_{\perp 2} & \kappa_{\theta\phi} \\ \kappa_{r\phi} & \kappa_{\theta\phi} & \kappa_{\|} \end{pmatrix}$$

# The Diffusion Tensor & An Overlooked Rotation?

## First Example: The Parker-Field

*(Effenberger et al. 2011)*

$\kappa_{\parallel} = 1,\ \kappa_{\perp 1} = 0.2,\ \kappa_{\perp 2} = 0.2$
$r = 10\text{AU},\ \varphi = \pi$

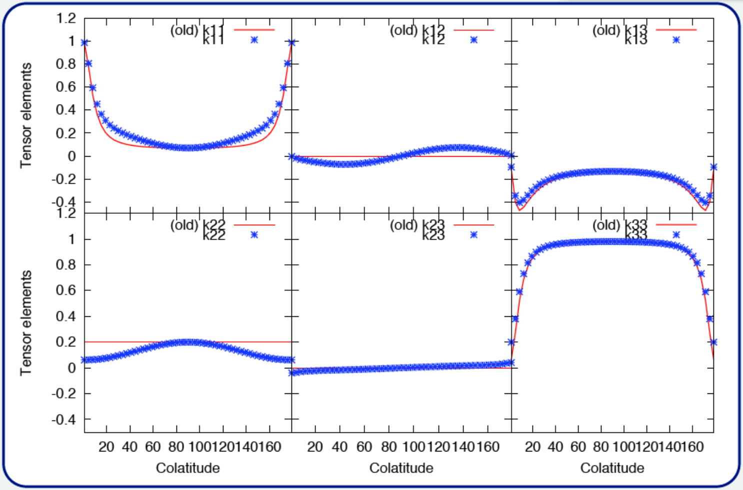


# The Diffusion Tensor & An Overlooked Rotation?

## First Example: The Parker-Field

*(Effenberger et al. 2011)*

$\kappa_{||} = 1,\ \kappa_{\perp 1} = 0.05,\ \kappa_{\perp 2} = 0.2$
$r = 10\mathrm{AU},\ \varphi = \pi$

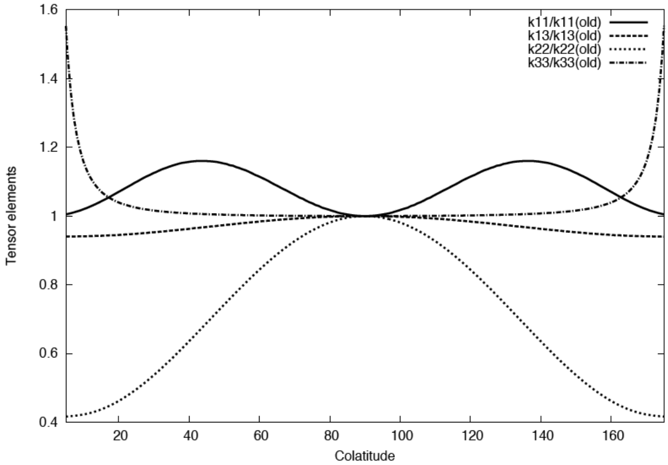


# The Diffusion Tensor & An Overlooked Rotation?

## First Example: The Parker-Field

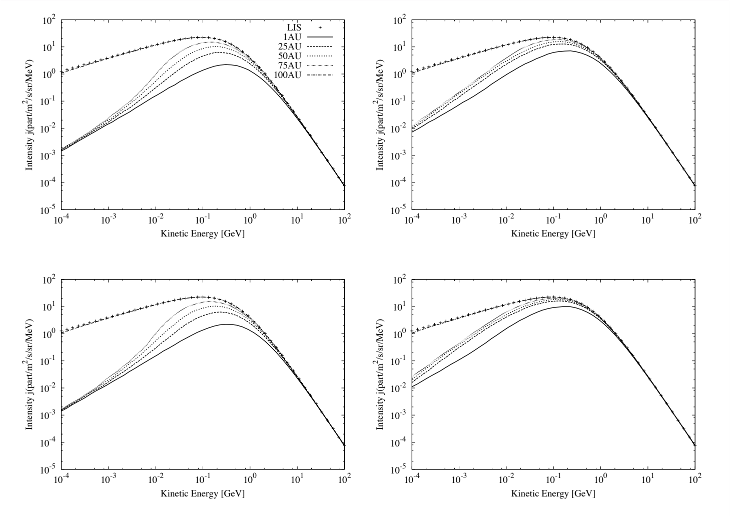
*(Effenberger et al. 2011)*

$\kappa_{\|} = 1,\ \kappa_{\perp 1} = 0.1,\ \kappa_{\perp 2} = 0.02$
$r = 10\mathrm{AU},\ \varphi = \pi$



# (Preliminary) Application to Modulated Spectra

*Burger & Hitge [2008]* new tensor

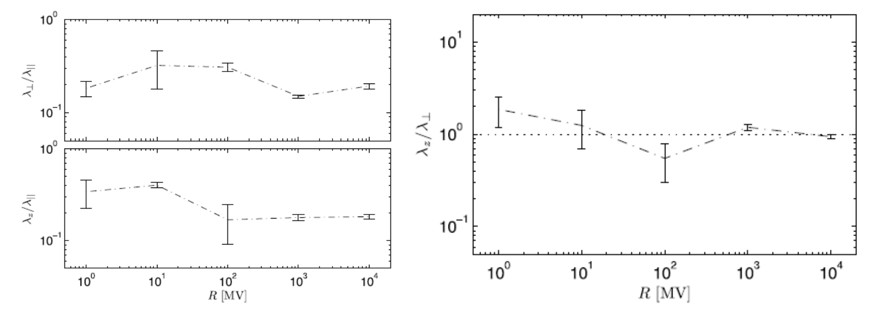



$\kappa_{\perp 2} = 5\kappa_{\perp 1}$

$\kappa_{\perp 2} = 10\kappa_{\perp 1}$

# Anisotropic perp. diffusion in simulations?

*Tautz et al. [2011]:*



# Anomalous Diffusion in the Heliosphere?

# Anomalous Diffusion

**What is anomalous diffusion?**

- normal (standard, Gaussian) diffusion:

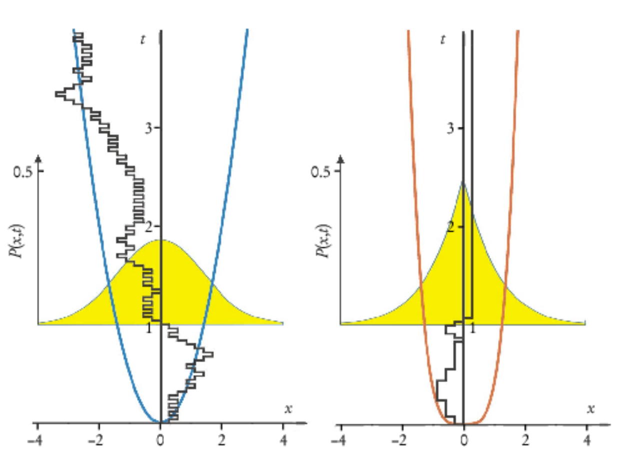
$$\langle (\Delta x)^2 \rangle = \kappa t$$

- anomalous (nonstandard, non-Gaussian) diffusion:

$$\langle (\Delta x)^2 \rangle = \kappa_\alpha t^\alpha; \qquad \begin{cases} \alpha < 1 & = \textit{subdiffusion} \\ \alpha > 1 & = \textit{superdiffusion} \end{cases}$$

**What are the consequences?**

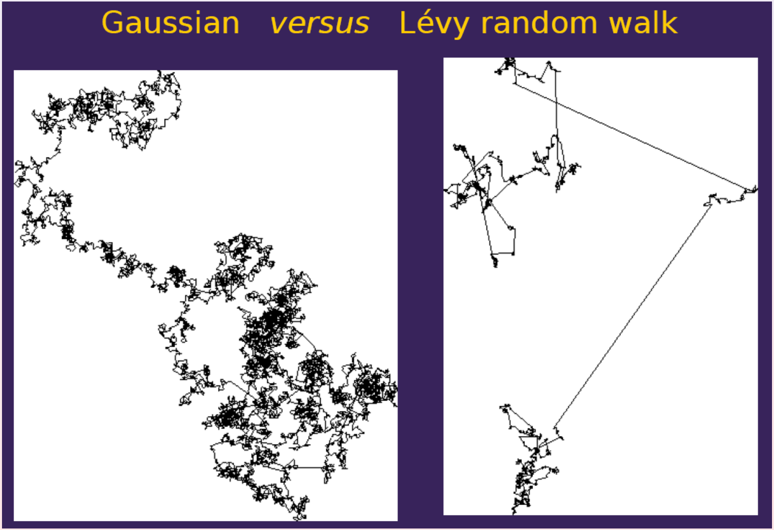
# Normal vs. anomalous (sub)diffusion




$\Delta t = const., \Delta x = const$ $\Delta t = \Psi(t), \Delta x = const$

normal diffusion $\longleftrightarrow$ CTRW

# Superdiffusion



# Observations of anomalous diffusion

Various authors have found **anomalous diffusion**:

- $\langle(\Delta z)^2\rangle = \kappa_{||} t^{\alpha}$ along $\vec{B}$:

  ACRs: $\alpha = 1.3$, i.e. superdiffusion (Perri & Zimbardo 2009)
  electrons: $\alpha \leq 1.4$, i.e. superdiffusion (Perri & Zimbardo 2008)
  electrons: $\alpha < 1$, i.e. subdiffusion (Albright et al. 2001)

- $\langle(\Delta x)^2\rangle = \kappa_{\perp} t^{\beta}$ perpendicular to $\vec{B}$:

  $\beta = 1/2$, i.e. subdiffusion (Kota & Jokipii 2000, Shalchi 2005, Webb et al. 2006)
  $\beta = 4/3$, i.e. superdiffusion (le Roux et al. 2010)

... at different energies and transport periods. In such cases the

**diffusion approximation (e.g. Parker equation) is invalid**

# A fractional (super)diffusion equation...

## 3 Übergang von parabolischen zu fraktionalen Differentialgleichungen

Die gewöhnlichen Diffusionsgleichungen erhält man, indem man das erste Ficksche Gesetz und die Kontinuitätsgleichung miteinander verknüpft. Man definiert den Fluss

$$\mathbf{j} = -D\nabla u \tag{14}$$

Dabei ist $D$ der Diffusionskoeffizient und $u$ eine beliebige Verteilung, zum Beispiel eine Konzentration oder Teilchenanzahl.
Setzt man Gleichung (14) in die Kontinuitätsgleichung

$$\nabla \cdot \mathbf{j} + \frac{\partial u}{\partial t} = 0 \tag{15}$$

ein, erhält man die bekannte Diffusionsgleichung

$$\frac{\partial u}{\partial t} = \nabla \cdot (D\nabla u) \tag{16}$$

Eine fraktionale Diffusionsgleichung erhält man, indem man den Nablaoperator im Fluss $\mathbf{j}$ auf fraktionale Ableitungen verallgemeinert. Das bedeutet der Fluss $\mathbf{j}$ lautet jetzt [Chaves, 1998]:

$$\mathbf{j} = -(D_+\nabla_+^{\alpha-1} - D_-\nabla_-^{\alpha-1})u \quad ; \quad 1 < \alpha \leq 2 \tag{17}$$

Dabei stehen $\nabla_+$ und $\nabla_-$ für den links- und rechtsfraktionalen Nablaoperator mit den entsprechenden Ableitungen nach den Koordinaten und die Vorfaktoren $D_+$ und $D_-$ sind die entsprechenden Diffusionskoeffizienten für die Streuung in positive und negative Richtung.
Unter der Annahme, dass $D_+$ und $D_-$ vom Ort unabhängig sind, kann man folgende fraktionale Diffusionsgleichung aufschreiben:

$$\frac{\partial u}{\partial t} = D_+\nabla \cdot (\nabla_+^{\alpha-1} u) - D_-\nabla \cdot (\nabla_-^{\alpha-1} u)$$

# … and its numerical solution for $\alpha = 4/3$ vs. $\alpha = 1$

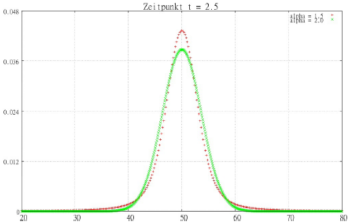



Abbildung 4: Vergleich von fraktionaler und normaler Diffusion. Die normale Diffusion startet schneller als die fraktionale.

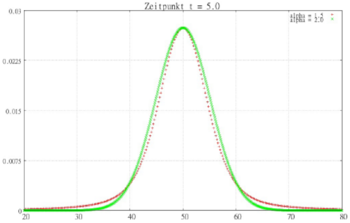



Abbildung 5: Der Diffusionskoeffizient der normalen Diffusion wurde so gewählt, dass die Lösungen der fraktionalen Diffusion und der normalen Diffusion zur Zeit $t = 5.0$ dieselbe Höhe haben.

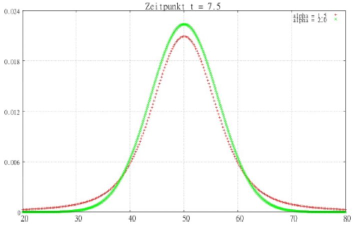



Abbildung 6: Die fraktionale Diffusion verläuft jetzt schneller als die normale.

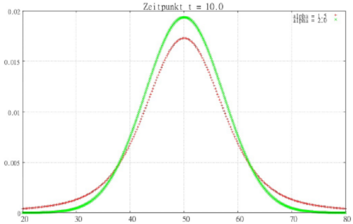



Abbildung 7: Endzeitpunkt der numerischen Lösung. Die fraktionale Diffusion verläuft mit der Zeit immer schneller.

# Résumé

# Résumé

**There is a lot of scope for refining and extending 'standard' theory.**

# ... and its numerical solution for $\alpha = 4/3$ vs. $\alpha = 1 \,\&\, \kappa(t)$

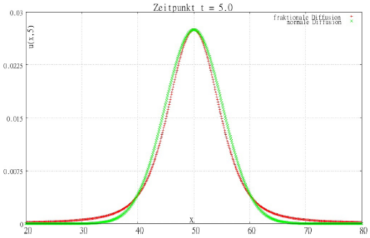



Abbildung 8: fraktionale Diffusion mit $\alpha = 1.5$, $D_- = D_+ = 1$ und normale Diffusion mit $D \sim t^{\frac{1}{3}}$

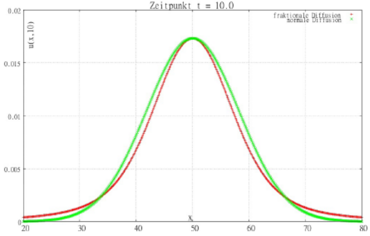



Abbildung 9: Parameter wie in 8 zu einem späteren Zeitpunkt

time-dependent normal diffusion can reproduce the same height but not the same width

# Numerical computation of fractional derivatives & integrals

Grünwald-Letnikov definition:

$$\frac{d^n f}{d[x-a]^n} = \lim_{N\to\infty} \left(\frac{x-a}{N}\right)^{-n} \sum_{j=0}^{N-1} \frac{\Gamma(j-n)}{\Gamma(-n)\Gamma(j+1)} f\left(x - j\frac{x-a}{N}\right)$$