Cosmic ray propagation in nonuniform turbulent magnetic fields

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References:

A new cosmic ray transport theory in partially turbulent space plasmas: Extending the quasilinear approach; RS, 2011, ApJ 732, 96

Cosmic ray transport in non-uniform magnetic fields: Consequences of gradient and curvature drifts; RS, F. Jenko, 2010, J. Plasma Phys. 76, 317

Focused acceleration of cosmic-ray particles in non-uniform magnetic fields; Y. Litvinenko, RS, 2011, ApJL 732, L31

1. Introduction

1.1. Fundamental questions

Which equations describe the dynamics of cosmic rays for given and specified electromagnetic fields (*test-particle approach*)?

How do the cosmic ray transport parameters depend on the statistical properties of the turbulent electromagnetic fields in space?

Under which conditions is cosmic ray transport diffusive? What are the differences of linear and nonlinear transport theories?

What causes perpendicular spatial diffusion? Not a single process: (1) scattering in perpendicular guiding center coordinates, (2) magnetic field-line random walk (compound diffusion), (3) gradient and curvature drift in nonuniform guide magnetic field.

Modern numerical cosmic ray transport codes such as GALPROP (Strong, Moskalenko et al.) investigate the diffusion-convection transport equation containing diffusion and convection terms in the particles′ momentum and space coordinates. Are all important physical effects represented?

No: The mirror forces in large-scale nonuniform guide magnetic fields provide additional transport effects: not only the well-established (from interplanetary observations) focused spatial convection but also focused acceleration.

1.2. History of diffusion-convection transport equations in uniform guide magnetic field

For isotropic part of gyrotropic particle phase space density $M(x_i, p, t)$, $x_i \in [x, y, z]$, because observed galactic cosmic rays are almost isotropic:

Parker (1963) equation:

$$\frac{\partial M}{\partial t} - S = \mathrm{div}\left[\kappa_{\mathrm{T}}\,\mathrm{gradM} - \tilde{\mathrm{U}}\mathrm{M}\right] + \frac{1}{\mathrm{p}^2}\frac{\partial}{\partial \mathrm{p}}\left[\frac{\mathrm{p}^3}{3}(\mathrm{div}\tilde{\mathrm{U}})\mathrm{M}\right] \qquad (1)$$

$\vec{U}$ **gas** velocity. Includes spatial diffusion, spatial convection and adiabatic deceleration/acceleration depending on $\mathrm{div}\,\vec{U} > 0$ or < 0, respectively (Fermi 1).

Skilling (1977) equation:

$$\frac{\partial M}{\partial t} - S = \mathrm{div}\left[\kappa_{\mathrm{T}}\,\mathrm{gradM} - \tilde{\mathrm{V}}\mathrm{M}\right] + \mathrm{p}^{-2}\frac{\partial}{\partial \mathrm{p}}\left[\mathrm{p}^2\mathrm{A}\frac{\partial \mathrm{M}}{\partial \mathrm{p}} + \frac{\mathrm{p}^3}{3}(\mathrm{div}\tilde{\mathrm{U}})\mathrm{M}\right] \qquad (2)$$

with effective cosmic ray bulk speed $\vec{V} = \vec{U} + H_c\vec{V}_A \neq \vec{U}$ with cross helicity $H_c \in [-1, 1]$ of scattering plasma waves and momentum diffusion (Fermi 2).

complete transport equation in uniform magnetic field (e.g. RS 2002)

$$\frac{\partial M}{\partial t} - S = \mathrm{div}\left[\kappa_{\mathrm{T}}\,\mathrm{gradM} - \tilde{\mathrm{V}}\mathrm{M}\right] - \frac{\mathrm{M}}{\mathrm{T_c}}$$
$$+ p^{-2}\frac{\partial}{\partial p}\left[p^2 A\frac{\partial M}{\partial p} + \frac{p^3}{3}(\mathrm{div}\tilde{\mathrm{V}})\mathrm{M} - \mathrm{p}^2\dot{\mathrm{p}}_{\mathrm{loss}}\mathrm{M}\right] \tag{3}$$

with continuous ($\dot{p}_{\mathrm{loss}}$) and catastrophic (T_c) loss processes.

Plus intrinsic relation (RS 1989, Dung and RS 1990) of parallel spatial diffusion and momentum diffusion

$$\kappa_{zz}A = G(H_c, \sigma_+, \sigma_-)V_A^2 p^2 \tag{4}$$

with magnetic helicities of forward (+) and backward (-) moving Alfven waves.

Also $V_i = V_i(H_c, \sigma_\pm)$.

GALPROP uses complete transport equation (good) but ignores intrinsic relation and helicity dependences.
E.g. in GALPROP $G = 1/9 = G(H_c = 0, \sigma_\pm = 0)$ and $\vec{V} = \vec{U}$ always! .

Remark: why should one use a numerical solution of the transport equation which can be fully solved analytically?

2. Fokker-Planck cosmic ray transport equation

We consider the transport of cosmic rays in a large-scale guide magnetic field $\vec{B}_0 = B_0 \vec{e}_z$, which is uniform on the scales of the cosmic ray particles gyradii $R_L = v/|\Omega|$. Because of the gyrorotation of the particles in the uniform magnetic field, one is not so much interested in their actual position as in the coordinates of the guiding center

$$\vec{X} = (X, Y, Z) = \vec{x} + \frac{\vec{v} \times \vec{e}_z}{\Omega} = \vec{x} + \frac{1}{\Omega} \begin{pmatrix} v_y \\ -v_x \\ 0 \end{pmatrix} \tag{5}$$

2.1. Vlasov equation

Transforming to the phase space variables (X, Y, z, p, μ, ϕ), the Vlasov (collision-free Boltzmann) equation reads (Hall and Sturrock 1968, Achatz et al. 1991)

$$\frac{\partial F}{\partial t} + v\mu \frac{\partial F}{\partial z} - \Omega \frac{\partial F}{\partial \phi} + p^{-2} \frac{\partial}{\partial y_\alpha} \left[p^2 h_\alpha(t) F \right] - Q_0(z, X, Y, p, \mu, \phi, t) = 0, \tag{6}$$

where we use the Einstein sum convention for indices. $y_\alpha \in [\mu, p, , \phi X, Y]$ represent the five phase space variables with non-vanishing stochastic fields $h_\alpha(t)$.

$$Q_0(z, X, Y, p, \mu, \phi, t) = S_0(z, X, Y, p, \mu, \phi, t) - \mathcal{N}_0 F - \mathcal{R}_0 F \tag{7}$$

accounts for sources and sinks (S_0) and the effects of the mirror force ($\mathcal{N}_0$) and momentum loss processes ($\mathcal{R}_0$), where the latter two operate on much longer spatial and time scales then the particle interactions with the stochastic fields. The equation of motion of charged particles provides with $\Omega = \frac{qB_0}{\gamma mc}$ and $\delta\vec{b} = \frac{\delta\vec{B}}{B_0}$

$$\frac{d\mu}{dt} = h_\mu(t) = \frac{\Omega}{v}\left(v_x\delta b_y - v_y\delta b_x\right) = \Omega\sqrt{1-\mu^2}\left(\cos\phi\delta b_y - \sin\phi\delta b_x\right), \quad (8)$$

and

$$\frac{d\phi}{dt} = -\Omega + h_\phi(t), \quad h_\phi(t) = -\Omega\delta b_z + \frac{\Omega\mu}{\sqrt{1-\mu^2}}\left(\cos\phi\delta b_x + \sin\phi\delta b_y\right). \quad (9)$$

with the two random forces $h_\mu(t)$ and $h_\phi(t)$. Eq. (9) also accounts for the regular force term $\dot{\phi} = -\Omega$ in Equation (6).
For the guiding center coordinates $X_i = [X, Y]$, $i, j = 1, 2$ we find

$$\frac{dX_i}{dt} = h_i(t) = v_z(t)\delta b_i(t) - v_i(t)\delta b_z(t), \quad (10)$$

2.2. Ensemble averaging

The distribution function F in Eq. (6) develops in an irregular way under the influence of the stochastic force fields $h_\alpha(t)$, but the detailed fluctuations are not of interest.

We seek an expectation value of F in terms of the statistical properties of $h_\alpha(t)$, so we consider an ensemble of distribution functions all beginning with identical values at time t_0. Let each of these functions be subject to a different member of an ensemble of realizations of $h_\alpha(t)$. At any time $t > t_0$, the various functions differ from each other, and we require an equation for $< F >$, the average of F over all members of the ensemble. With $F =< F > +\delta F$ and the operator $\mathcal{L}_0 = \partial_t + \ v\mu\partial_Z - \ \Omega\partial_\phi$, Eq. (6) for neglected electric field fluctuations ($h_p = 0$) reads

$$\mathcal{L}_0 < F > +\mathcal{L}_0\delta F + \frac{\partial}{\partial y_\alpha}\left[h_\alpha(t) < F >\right] + \frac{\partial}{\partial y_\alpha}\left[h_\alpha(t)\delta F\right] - Q_0 = 0 \qquad (11)$$

Ensemble-averaging Eq. (11) using $< h_\alpha(t) >=< \delta F >= 0$ yields the desired kinetic equation for $< F >$:

$$\mathcal{L}_0 < F > -Q_0(z, X, Y, p, \mu, \phi, t) = -\frac{\partial}{\partial y_\alpha}\left[< h_\alpha(t)\delta F >\right], \qquad (12)$$

Substracting Eq. (11) from Eq. (12) gives the equation for the deviation

$$\mathcal{L}_0\delta F = -h_\alpha(t)\frac{\partial < F >}{\partial y_\alpha} - h_\alpha(t)\frac{\partial \delta F}{\partial y_\alpha} + < h_\alpha(t)\frac{\partial \delta F}{\partial y_\alpha} > \qquad (13)$$

With the inverted time-integration operator $\mathcal{L}_0^{-1}$ the formal solution of Eq. (13) provides for the ensemble average on the right hand side of Eq. (12) is

$$< h_\alpha(t)\delta F >= - < h_\alpha(t)\mathcal{L}_0^{-1} h_\sigma(t) \frac{\partial < F >}{\partial y_\sigma} > - < h_\alpha(t)\mathcal{L}_0^{-1} \frac{\partial (h_\sigma(t)\delta F)}{\partial y_\sigma} >, \tag{14}$$

which is an integral-type equation.

2.3. Weak turbulence or quasilinear approximation

Keeping only first-order terms in the fluctuating quantities δF and $h_\alpha(t)$, which are assumed to be small (weak-turbulence assumption or quasilinear approximation) then yields

$$< h_\alpha(t)\delta F >\simeq - < h_\alpha(t)\mathcal{L}_0^{-1} h_\sigma(t) \frac{\partial < F >}{\partial y_\sigma} >\simeq< h_\alpha(t)\mathcal{L}_0^{-1} h_\sigma(t) > \frac{\partial < F >}{\partial y_\sigma}, \tag{15}$$

where we follow the arguments of Hall and Sturrock (1968) and Achatz et al. (1991), that for weak turbulence $\partial < F > /\partial y_\sigma$ varies only negligibly over the time-integration interval. We arrive at the kinetic equation

$$\mathcal{L}_0 < F > -Q_0(z, X, Y, p, \mu, \phi, t) = -\frac{\partial}{\partial y_\alpha} T_{\alpha\sigma} \frac{\partial < F >}{\partial y_\sigma} \tag{16}$$

with the full Fokker-Planck coefficients

$$T_{\alpha\sigma} =< h_\alpha(t)\mathcal{L}_0^{-1}h_\sigma(t) > \tag{17}$$

The quasilinear time-integration operator is obtained by integrating along the characteristics of the operator $\mathcal{L}_0$ (Achatz et al. 1991), which is the unperturbed gyrocenter orbit in the uniform magnetic field $(X_u = X,\ Y_u = Y,\ Z_u = Z + v\mu(u-t),\ p_u = p,\ \mu_u = \mu,\ \phi_u = \phi - \Omega(u-t))$, so that

$$\mathcal{L}_0^{-1}h_\sigma(t) = \int_{t_0}^{t} du\, h_\sigma(u) = \int d^3k \int_{t_0}^{t} du\, H_\sigma(\vec{k},u)e^{\imath\vec{k}\cdot\vec{x}(u)}, \tag{18}$$

after Fourier transforming the stochastic force in space.
Consequently,

$$\begin{aligned} T^{\rm ql}_{\alpha\sigma} &=< h_\alpha(t)\int_{t_0}^{t} du\, h_\sigma(u) > \\ &= \int d^3k < h_\alpha(t)\int_{t_0}^{t} du\ H_\sigma(\vec{k},u)e^{\imath\vec{k}\cdot\vec{X}} \\ &\times \exp\left[\imath v\mu k_\parallel(u-t) + \imath\frac{k_\perp v\sqrt{1-\mu^2}}{\Omega}\sin(\psi-\phi-\Omega(t-u))\right] > \end{aligned} \tag{19}$$

2.4. More general particle orbits

Deviations from the unperturbed particle orbits resulting from the higher order term in Eq. (14) in magnetic turbulence affect the gyrophase $\phi(t)$ and the pitch-angle $\mu(t)$. Here we consider a more general class of particle orbits with deviations of the gyrophase given by

$$X_s = X,\ Y_s = Y,\ Z_s = Z + v\mu(s-t),\ p_s = p,$$
$$\mu_s = \mu,\ \phi_s = \phi - \Omega(s-t) + \delta\phi(t-s), \tag{20}$$

that contains the additional arbitrary gyrophase variation $\delta\phi(t-s)$, with $\delta\phi = 0$ for $s = t$, affecting in particular the perpendicular transport of cosmic ray particles. The quasilinear orbits are reproduced by setting $\delta\phi = 0$. In this case the full Fokker-Planck coefficients (17) become

$$T_{\alpha\sigma} = \int d^3k < h_\alpha(t) \int_{t_0}^{t} ds\, H_\sigma(\vec{k}, s) e^{\imath\vec{k}\cdot\vec{X} + \imath v\mu k_\parallel (s-t)}$$
$$\times \exp\left[\imath k_\perp v\sqrt{1-\mu^2}\int^{s} dw\, \cos\left(\psi - \phi + \Omega(w-t) - \delta\phi(t-w)\right)\right] > \tag{21}$$

Introducing the variables $x_\nu \in [X, Y, \mu]$ and the explicit form of the operator $\mathcal{L}_0$ the kinetic equation (16) reads

$$\partial_t < F > +v\mu\partial_z < F > -\Omega\partial_\phi < F > -Q_0(z, X, Y, p, \mu, \phi, t) =$$
$$- \frac{\partial}{\partial x_\alpha} T_{\alpha\sigma} \frac{\partial < F >}{\partial x_\sigma} - \frac{\partial}{\partial \phi} T_{\phi\sigma} \frac{\partial < F >}{\partial x_\sigma} - \frac{\partial}{\partial x_\alpha} T_{\alpha\phi} \frac{\partial < F >}{\partial \phi} \tag{22}$$

2.5. Gyrotropic distribution functions

We now employ the *small Larmor radius approximation* (Chew et al. 1956, Kennel and Engelmann 1962) that in the presence of the guide magnetic field all changes are considered small over space scales comparable with the particle Larmor radii or time scales comparable with typical gyroperiods. Therefore the Larmor radius and gyroperiod are convenient small expansion parameters. The Larmor orbiting of particles is so rapid that all inhomogeneities in the ϕ-distribution of particles are smoothed out on the macroscopic scale, and the distribution functions are independent of ϕ to lowest order. With the expansion

$$< F >= f + \frac{F_1}{\Omega} \tag{23}$$

inserted in Eq. (22) we then find to lowest order

$$\frac{\partial f}{\partial \phi} = 0 \tag{24}$$

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Thus the lowest-order distribution function is independent of the gyrophase ϕ. To find the spatial and time dependence of f we go to next order giving

$$\partial_t f + v\mu\partial_z f - \partial_\phi \left[F_1 - T_{\phi\sigma} \frac{\partial f}{\partial x_\sigma} \right] - Q_0(z, X, Y, p, \mu, \phi, t) = -\frac{\partial}{\partial x_\alpha} T_{\alpha\sigma} \frac{\partial f}{\partial x_\sigma} \tag{25}$$

The physical requirement that f and F_1 be periodic in ϕ then removes the third term on the left hand side when averaging Eq. (25) from 0 to 2π in ϕ, leading to the Larmor-phase-averaged Fokker-Planck equation

$$\partial_t f + v\mu\partial_z f - Q(z, X, Y, p, \mu, t) = -\frac{\partial}{\partial x_\alpha} D_{\alpha\sigma} \frac{\partial f}{\partial x_\sigma} - \tag{26}$$

with the gyro-averaged source term

$$Q(z, X, Y, p, \mu, t) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, Q_0(z, X, Y, p, \mu, \phi, t), \tag{27}$$

and the gyro-averaged Fokker-Planck coefficients

$$D_{\alpha\sigma} = \Re \frac{1}{2\pi} \int_0^{2\pi} d\phi T_{\alpha\sigma} = \Re \frac{1}{2\pi} \int_0^{2\pi} d\phi < h_\alpha(t) \int_{t_0}^{t} ds \, h_\sigma^*(s) >, \tag{28}$$

where we replaced $h_\sigma(t) = h_\sigma^*(t)$ by its complex conjugate because the stochastic forces are real-valued quantities.

2.6. Gyro-averaged Fokker-Planck equation

The full Fokker-Planck equation (26) including electric field fluctuations for $f(X, Y, z, p, \mu, t)$ then reads (Schlickeiser and Jenko 2010)

$$\frac{\partial f}{\partial t} + \ v\mu \frac{\partial f}{\partial z} + \mathcal{N} f + \mathcal{R} f - S(z, X, Y, p, \mu, t) = p^{-2} \partial_\nu \left(p^2 D_{\nu\sigma} \partial_\sigma f \right], \quad (29)$$

where

$$\mathcal{N} f = \frac{v}{2L_3} \frac{\partial}{\partial \mu} \left[\left(1 - \mu^2\right) f \right] + \frac{\epsilon_a v R_L (1 - \mu^2)}{2L_2} \frac{\partial f}{\partial X} - \frac{\epsilon_a v R_L (1 - \mu^2)}{2L_1} \frac{\partial f}{\partial Y} \quad (30)$$

accounts for the effects of the mirror force in the large spatial gradients ($L_i^{-1} = -\partial_{x_i} \ln B_0$) of the guide field, and

$$\mathcal{R} f = p^{-2} \partial_p \left[p^2 \dot{p}_{\text{loss}} f \right] + \frac{f}{T_c} \quad (31)$$

represent continous ($\dot{p}_{\text{loss}}$) and catastrophic (T_c) momentum losses of particles. $S(z, X, Y, p, t)$ represents additional sources and sinks of particles.
In Eq. (29) we use the short notation $\partial_\nu = (\partial / \partial x_\nu)$, where $x_{\nu,\sigma} \in [\mu, p, X, y]$ represent the four phase space variables μ, p, X, Y with non-vanishing stochastic fields $h_\nu(t)$. Therefore the terms on the right-hand side of Eq. (29) in general represent 16 different Fokker-Planck terms; however, depending on the specific type of turbulence considered, not all of them are non-zero, and some of them are much larger than others.

3. Incompressible magnetic turbulence

Restricting our analysis to incompressible magnetic turbulence ($\delta b_z = 0$), the equation of motion of the guiding center simplifies to $h_i(t) = v\mu\delta b_i(t)$.

3.1. Step 1: Quasi-stationary turbulence

As first assumption we use the *quasi-stationary turbulence condition* that the correlation function $< h^*_\nu(t)h_\sigma(s) >$ in Eq. (28) depends only on the absolute value of the time difference $|t-s| = |\tau|$ so that with the substitution $s = t-\tau$ we find for Eq. (28)

$$D_{\nu\sigma} == \Re\frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^{t-t_0} ds \; < h_\nu(t)h^*_\sigma(t-\tau) > \tag{32}$$

3.2. Step 2: Existence of finite decorrelation time

If there exists a *finite decorrelation time* t_c such that the correlation functions $< h_\nu(t)h^*_\sigma(t-\tau) > \to 0$ fall to a negligible magnitude for $\tau \to \infty$, this allows us to replace the upper integration boundary in the τ-integral by infinity so that

$$D_{\nu\sigma} = \Re\frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^{\infty} d\tau \; < h_\nu(t)h^*_\sigma(t-\tau) > . \tag{33}$$

The two assumptions of quasi-stationary turbulence and the existence of a finite turbulence decorrelation time t_c guarantee diffusive transport behaviour.

3.3. Step 3: Homogeneous turbulence

With the corresponding Fourier transform in space of $h_\nu(t)$ the gyro-averaged Fokker-Planck coefficients (32) read

$$D_{\nu\sigma} = \Re \frac{1}{2\pi} \int_0^{2\pi} d\phi \int d^3k' \int d^3k \, e^{\imath(\vec{k}' - \vec{k})\cdot \vec{X}} \int_0^\infty ds \, < H_\nu(\vec{k}', t) H^*_\sigma(\vec{k}, s)$$
$$\times \, e^{-\imath v \mu k_\parallel (s-t) - \imath v \sqrt{1-\mu^2}\left(k_\perp \int^s dw \, \cos(\psi - \phi + \Omega(w-t) - \delta\phi(t-w)) + k'_\perp \frac{\sin(\phi - \psi')}{\Omega}\right)} > \tag{34}$$

As third assumption we use that *the turbulent magnetic fields are homogenously distributed*, meaning that independent from the actual position of the gyrocenter at time t the particles are subject to turbulence realizations with the same statistical properties. This allows us to average Eq. (34) over the spatial position of the guiding center using

$$\frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d^3X \, e^{i(\vec{k}' - \vec{k})\cdot\vec{X}} = \delta(\vec{k}' - \vec{k}), \tag{35}$$

yielding

$$D_{\nu\sigma} = \Re \frac{1}{2\pi} \int_0^{2\pi} d\phi \int d^3k \int_0^\infty d\tau \, < H_\nu(\vec{k}, t) H^*_\sigma(\vec{k}, t-\tau) e^{\imath v \mu k_\parallel \tau}$$
$$\times \, e^{-\imath k_\perp v \sqrt{1-\mu^2}\left(\int^{t-\tau} dw \, \cos(\psi - \phi + \Omega(w-t) - \delta\phi(t-w)) + \frac{\sin(\phi-\psi)}{\Omega}\right)} > \tag{36}$$

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3.4. Step 4: Corrsin-type assumption on the nature of generalized orbits

As fourth assumption we here restrict our analysis to particle orbits where $\delta\phi(w)$ is independent from the fluctuating fields, so that the ensemble averaging in Eq. (36) involves only the 2nd order correlation functions of the stochastic fields. This is a severe restriction, and basically corresponds to the Corrsin independence hypothesis (Corrsin 1959, Salu and Montgomery, McComb 1990). With $\xi = t - w$ and the abbreviation $G(\xi) = \Omega\xi + \delta\phi(\xi)$ the Fokker-Planck coefficients (36) then are

$$D_{\nu\sigma} = \Re\frac{1}{2\pi}\int_0^{2\pi} d\phi \int d^3k \int_0^{\infty} d\tau < H_\nu(\vec{k},t)H^*_\sigma(\vec{k},t-\tau) > \\ \times e^{\imath v\mu k_\parallel \tau + \imath k_\perp v\sqrt{1-\mu^2}\left(\int^\tau d\xi \cos(\phi-\psi+G(\xi)) - \frac{\sin(\phi-\psi)}{\Omega}\right)}, \quad (37)$$

3.5. Step 5: Axisymmetric turbulence

An enormous simplification results for axisymmetric turbulence

$$< b_i(\vec{k},t)b^*_j(\vec{k},t-\tau) >= P_{ij}(\vec{k},\tau) = P_{ij}(k_\parallel, k_\perp, \tau), \quad (38)$$

independent of the wave phase ψ. The only remaining Fokker-Planck coefficients for magnetic turbulence then are

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$$D_{ij} = \Re\, 2\pi v^2\mu^2 \int_{-\infty}^{\infty} dk_\parallel \int_0^\infty dk_\perp\, k_\perp \int_0^\infty d\tau\, e^{\imath v\mu k_\parallel \tau} P_{ij}(k_\parallel, k_\perp, \tau) J_0(Z), \tag{39}$$

and

$$\begin{aligned} D_{\mu\mu} = \Re\, \pi\Omega^2(1-\mu^2) \int_{-\infty}^{\infty} dk_\parallel \int_0^\infty dk_\perp\, k_\perp \int_0^{t-t_0} d\tau\, J_0(Z) \\ \times \left[e^{\imath(v\mu k_\parallel \tau + G(\tau))} P_{LL}(k_\parallel, k_\perp, \tau) + e^{\imath(v\mu k_\parallel \tau - G(\tau))} P_{RR}(k_\parallel, k_\perp, \tau)\right] \end{aligned} \tag{40}$$

in terms of the left-handed and right-handed polarized stochastic magnetic field components

$$2P_{LL} = P_{xx} + P_{yy} + \imath P_{yx} - \imath P_{xy}, \quad 2P_{RR} = P_{xx} + P_{yy} + \imath P_{xy} - \imath P_{yx}, \tag{41}$$

and

$$Z = k_\perp v\sqrt{1-\mu^2}\left[\left(\int^\tau d\xi\ \cos(G(\xi))\right)^2 + \left(\frac{1}{\Omega} + \int^\tau d\xi\ \sin(G(\xi))\right)^2\right]^{1/2} \tag{42}$$

3.6. Quasilinear limit

The quasilinear approximation to the particle orbits is recovered by setting $\delta\phi = 0$ providing $G(\tau) = \Omega\tau$, so that Eq. (42) becomes

$$Z = Z_0 = \frac{2k_\perp v\sqrt{1-\mu^2}}{\Omega}|\sin(\frac{\Omega\tau}{2})| \tag{43}$$

The Fokker-Planck coefficients then become

$$D_{ij}^{QL} = \Re\, 2\pi v^2\mu^2 \int_0^\infty d\tau \int_{-\infty}^\infty dk_\parallel \int_0^\infty dk_\perp\, k_\perp e^{\imath k_\parallel v_\parallel \tau} J_0\left(Z_0\right) P_{ij}(k_\parallel, k_\perp, \tau), \tag{44}$$

and

$$D_{\mu\mu}^{QL} = \Re\, \pi\Omega^2(1-\mu^2) \int_0^\infty d\tau \int_{-\infty}^\infty dk_\parallel \int_0^\infty dk_\perp\, k_\perp J_0(Z_0)$$
$$\times \left[e^{\imath(v\mu k_\parallel + \Omega)\tau}(P_{LL}(k_\parallel, k_\perp, \tau) + e^{\imath(v\mu k_\parallel - \Omega)\tau}(P_{RR}(k_\parallel, k_\perp, \tau)\right] \tag{45}$$

Note that these quasilinear Fokker-Planck coefficients in axisymmetric turbulence **no longer involve infinite sums of products of Bessel functions**.

3.7. Strict upper limits

Independent of the actual cosmic ray phase orbit $\delta\phi(\tau)$ we obtain with $J_0(A) \leq 1$ and $\cos(x) \leq 1$

$$D_{ij} < D_{ij}^{\max} = \frac{v^2 \mu^2 \delta b_{ij}^2}{\gamma} \tag{46}$$

and

$$D_{\mu\mu} < D_{\mu\mu}^{\max} = \frac{\Omega^2(1-\mu^2)\left[\delta b_{LL}^2 + \delta b_{RR}^2\right]}{2\gamma} = \frac{\Omega^2(1-\mu^2)\left[\delta b_{xx}^2 + \delta b_{yy}^2\right]}{2\gamma}, \tag{47}$$

using an exponential magnetic field fluctuation decorrelation time $t_c = \gamma^{-1}$

$$P_{ij}(\vec{k}, \tau) = P_{ij}^0(\vec{k}) e^{-\gamma\tau}, \tag{48}$$

Correspondingly, with the diffusion approximation (neglecting the influence of the mirror force) we find the upper limits for the perpendicular spatial diffusion coefficients

$$\kappa_{ij} < \kappa_{ij}^{\max} = \frac{v^2 \delta b_{ij}^2}{3\gamma} \tag{49}$$

and the lower limit for the parallel spatial diffusion coefficient

$$\kappa_\parallel > \kappa_\parallel^{\min} = \frac{\gamma v^2}{3\Omega^2\left[\delta b_{xx}^2 + \delta b_{yy}^2\right]}, \tag{50}$$

implying the general relation

$$\kappa_\parallel^{\min}\left[\kappa_{XX}^{\max} + \kappa_{YY}^{\max}\right] = \left(\frac{vR_L}{3}\right)^2 \tag{51}$$

In terms of the associated mean free paths the last relation reads

$$\lambda_\parallel^{\min}\left[\lambda_{XX}^{\max} + \lambda_{YY}^{\max}\right] = R_L^2 \tag{52}$$

If the parallel diffusion is limited by Bohm diffusion ($\lambda_\parallel^{\min} \geq R_L$), the relation (52) dictates that the sum of the perpendicular mean free paths

$$\lambda_{XX}^{\max} + \lambda_{YY}^{\max} \leq R_L \tag{53}$$

has to be smaller than R_L, in agreement with the theorem of reduced dimensionality. We note that this derivation does not account for possible additional field-line random walk and gradient and curvature drifts.

For compressive magnetic turbulence (Casanova and RS 2011)

$$\lambda_\parallel^{\min}\left[\lambda_{XX}^{\max} + \lambda_{YY}^{\max}\right] = 2R_L^2 \tag{54}$$

3.8. Relation to alternative nonlinear transport theory

In several alternative treatments of cosmic ray transport (e.g. Matthaeus et al. 2003; Shalchi 2006, 2009; Le Roux et al. 2010), the ensemble averaging in Eq. (36) for the Fokker-Planck coefficients is done differently, anticipating the expected diffusive motion of particles (standard NLGC theory). Adopting that approach here means

$$
\begin{aligned}
&< H_i^*(\vec{k},t)H_j(\vec{k},t-\tau)e^{\imath k_\parallel v\mu} \\
&\times e^{\imath k_\parallel v\mu - \imath k_\perp v\sqrt{1-\mu^2}\left(\int^\tau dw\, \cos(\phi-\psi+\Omega(t-w)+\delta\phi(t-w))+\frac{\sin(\phi-\psi)}{\Omega}\right)} > \\
&=< H_i^*(\vec{k},t)H_j(\vec{k},t-\tau) > \exp[-\frac{v^2\tau}{3\kappa_\parallel} - \kappa_\parallel k_\parallel^2\tau - \sum_{i,j} D_{ij}k_ik_j\tau], \qquad (55)
\end{aligned}
$$

where $\kappa_\parallel$ denotes the parallel spatial diffusion coefficient. As a consequence, we obtain a nonlinear formula for e.g. the perpendicular Fokker-Planck coefficients

$$
D_{ij} = \Re\, v^2\mu^2 \int d^3k \int_0^{t-t_0} d\tau P_{ij}(\vec{k},\tau) \exp\left[-\frac{v^2\tau}{3\kappa_\parallel} - \kappa_\parallel k_\parallel^2\tau - \sum_{i,j} D_{ij}k_ik_j\tau\right] \qquad (56)
$$

Adopting again the exponential magnetic field fluctuation decorrelation time $P_{ij}(\vec{k},\tau) = P_{ij}^0(\vec{k})e^{-\gamma\tau}$, we then obtain in the diffusion limit $t - t_0 \gg t_c$

$$D_{ij} = \Re\, v^2\mu^2 \int_0^\infty d\tau \int_{-\infty}^{\infty} d^3k\; P^0_{ij}(\vec{k}) \exp\left[-\gamma\tau - \frac{v^2\tau}{3\kappa_\parallel} - \kappa_\parallel k_\parallel^2 \tau - \sum_{i,j} D_{ij} k_i k_j \tau\right]$$

$$= v^2\mu^2 \int_{-\infty}^{\infty} d^3k\, \frac{P^0_{ij}(\vec{k})}{\gamma + \frac{v^2}{3\kappa_\parallel} + \kappa_\parallel k_\parallel^2 + \sum_{i,j} D_{ij} k_i k_j}, \qquad (57)$$

relating nonlinearly the perpendicular Fokker-Planck coefficients to the parallel spatial diffusion coefficient.

Improved NLGC theory (Shalchi 2010, 2011): replace ensemble-averaging of turbulent fields by cosmic ray particle average using the Fokker-Planck equation but see criticism by Lerche and Tautz (2011, PoP 18, 082305).

4. Modified diffusion-convection transport equation

For MHD turbulence the fluctuating electric fields are much smaller than the fluctuating magnetic fields ($\delta E \ll \delta B$). In this case the gyrotropic particle phase space distribution function $f(X,Y,z,p,\mu,t)$ due to dominating pitch-angle diffusion adjusts very quickly to a quasi-equilibrium through pitch-angle diffusion which is close to the isotropic equilibrium distribution $M(X,Y,z,p,t)$. For axisymmetric incompressible turbulence, only D_{XX}, D_{YY}, $D_{\mu\mu}$, $D_{\mu p}$ and D_{pp} are non-vanishing. In this case the diffusion approximation (Jokipii 1966, Hasselmann and Wibberenz 1968, Skilling 1975, Schlickeiser 1989) yields

$$\frac{\partial M}{\partial t} - S + \frac{\partial}{\partial X}\left(\left[1-\frac{vK_1}{8L_3}\right]\frac{\epsilon_a v r_L M}{3L_2}\right) - \frac{\partial}{\partial Y}\left(\left[1-\frac{vK_1}{8L_3}\right]\frac{\epsilon_a v r_L M}{3L_1}\right)$$
$$+\frac{\partial}{\partial z}\left(\frac{\kappa_{zz}}{L_3}M\right) + \frac{M}{T_c} + \frac{1}{p^2}\frac{\partial}{\partial p}\left(\left[p^2\dot{p}_{\rm loss} - \frac{a_{zp}p^2}{L_3}\right]M\right)$$
$$= \begin{pmatrix} \partial_X \\ \partial_Y \\ \partial_z \\ p^{-2}\partial_p p^2 \end{pmatrix} \cdot \begin{pmatrix} \kappa_{XX} & \kappa_{YX} & \kappa_{zX} & a_{Xp} \\ \kappa_{YX} & \kappa_{YY} & \kappa_{zY} & a_{Yp} \\ \kappa_{zX} & \kappa_{zY} & \kappa_{zz} & a_{zp} \\ -a_{Xp} & -a_{Yp} & -a_{zp} & A \end{pmatrix} \begin{pmatrix} \partial_X M \\ \partial_Y M \\ \partial_z M \\ \partial_p M \end{pmatrix} \tag{58}$$

with the pitch-angle averaged transport parameters

$$A = \frac{1}{2}\int_{-1}^{1} d\mu\left[D_{pp}(\mu) - \frac{D_{\mu p}^2(\mu)}{D_{\mu\mu}(\mu)}\right], \tag{59}$$

$$\kappa_{zz} = \frac{v^2 K_0}{8} = \frac{v^2}{8} \int_{-1}^{1} d\mu \frac{(1-\mu^2)^2}{D_{\mu\mu}(\mu)}, \tag{60}$$

$$\kappa_{XX} = \frac{1}{2} \int_{-1}^{1} d\mu D_{XX}(\mu) + \frac{v^2 r_L^2}{72 L_2^2} \int_{-1}^{1} d\mu \frac{\mu^2(1-\mu^2)^2}{D_{\mu\mu}(\mu)}, \tag{61}$$

$$\kappa_{YY} = \frac{1}{2} \int_{-1}^{1} d\mu D_{XX}(\mu) + \frac{v^2 r_L^2}{72 L_1^2} \int_{-1}^{1} d\mu \frac{\mu^2(1-\mu^2)^2}{D_{\mu\mu}(\mu)}, \tag{62}$$

$$\kappa_{YX} = -\frac{v^2 r_L^2}{72 L_1 L_2} \int_{-1}^{1} d\mu \frac{\mu^2(1-\mu^2)^2}{D_{\mu\mu}(\mu)}, \tag{63}$$

$$\kappa_{zX} = -\frac{\epsilon_a v^2 r_L K_1}{24 L_2} = -\frac{\epsilon_a v^2 r_L}{24 L_2} \int_{-1}^{1} d\mu \frac{\mu(1-\mu^2)^2}{D_{\mu\mu}(\mu)}, \tag{64}$$

$$\kappa_{zY} = \frac{\epsilon_a v^2 r_L}{24 L_1} \int_{-1}^{1} d\mu \frac{\mu(1-\mu^2)^2}{D_{\mu\mu}(\mu)}, \tag{65}$$

$$a_{Xp} = -\frac{\epsilon_a v r_L}{12 L_2} \int_{-1}^{1} d\mu \frac{\mu(1-\mu^2) D_{\mu p}(\mu)}{D_{\mu\mu}(\mu)}, \tag{66}$$

$$a_{Yp} = \frac{\epsilon_a v r_L}{12 L_1} \int_{-1}^{1} d\mu \frac{\mu(1-\mu^2) D_{\mu p}(\mu)}{D_{\mu\mu}(\mu)}, \tag{67}$$

$$a_{zp} = \frac{v}{4} \int_{-1}^{1} d\mu \frac{(1-\mu^2) D_{\mu p}(\mu)}{D_{\mu\mu}(\mu)}, \tag{68}$$

- The modified diffusion-convection transport equation (58) is more involved than anything solved in the GALPROP or other cosmic ray transport codes. In particular, it implies new physical processes such as focused acceleration.

- Most of the simplifying assumptions made are concerned with the calculation of the Fokker-Planck coefficients.

- The origin of each transport term is clearly identified and can be related to the statisticsl properties of the turbulent electromagnetic fields and/or the spatial gradients of the guide magnetic field.

- The spatial diffusion coefficients κ_{zz}, κ_{XX}, κ_{YX} and κ_{YY} are independent from the charge-sign of the cosmic ray particle.

- At the contrast, the convective transport terms a_{Xp} and a_{Yp} as well as the spatial diffusion coefficients κ_{zX} and κ_{zY} exhibit the cosmic ray charge-sign dependence.

- For infinitely large perpendicular scale length ($L_1 = L_2 = \infty$), Eq. (58) reduces to the focused diffusion-convection transport equation, considered in the next section in more detail.

- For turbulence geometries (e.g. slab Alfven waves) with vanishing D_{XX} and D_{YY} all perpendicular spatial diffusion coefficients are caused by the non-vanishing gradient and curvature drift terms combined with different averages involving $D_{\mu\mu}^{-1}$.

4.1. Perpendicular spatial diffusion from gradient and curvature drifts

For a specific (symmetric in μ) choice of the pitch-angle Fokker-Planck coefficient $D_{\mu\mu} = \frac{v^2}{2(2-s)(4-s)\kappa_{zz}}|\mu|^{s-1}(1-\mu^2)$ we obtain $\kappa_{zX} = \kappa_{zY} = 0$, and for the ratios of the remaining perpendicular to parallel spatial diffusion coefficients and mean free paths

$$\frac{\kappa_{XX}}{\kappa_{zz}} = \frac{\lambda_{XX}}{\lambda_{zz}} = \frac{2-s}{6-s}\left(\frac{r_L}{9L_2}\right)^2 = \frac{2-s}{36(6-s)}\left(\frac{\partial r_L}{\partial Y}\right)^2, \qquad (69)$$

$$\frac{\kappa_{YY}}{\kappa_{zz}} = \frac{\lambda_{YY}}{\lambda_{zz}} = \frac{2-s}{6-s}\left(\frac{r_L}{3L_1}\right)^2 = \frac{2-s}{9(6-s)}\left(\frac{\partial r_L}{\partial X}\right)^2, \qquad (70)$$

and

$$\frac{\kappa_{YX}}{\kappa_{zz}} = \frac{\lambda_{YX}}{\lambda_{zz}} = -\frac{2-s}{6-s}\left(\frac{r_L}{3L_1}\right)\left(\frac{r_L}{3L_2}\right) = -\frac{2-s}{9(6-s)}\frac{\partial r_L}{\partial X}\frac{\partial r_L}{\partial Y}, \qquad (71)$$

involving the spatial first derivatives of the non-constant Larmor radius in the non-uniform magnetic field. Due to the Larmor radius dependence, the ratios of perpendicular to parallel spatial diffusion coefficients increase proportional ($\propto (p/|q_a|)^2$) to the square of the particle rigidity. Moreover, cosmic rays with the same rigidity value have the same ratios of perpendicular to parallel spatial diffusion coefficients.

5. Focused diffusion-convection transport equation

For $L_1 = L_2 = \infty$ and $D_{XX} = D_{YY} = 0$ the modified diffusion-convection equation (58) reduces to the focused diffusion-convection transport equation ($L_3 = L$)

$$\frac{\partial M}{\partial t} - S + \frac{\partial}{\partial z}\left(\frac{\kappa_{zz}}{L} M\right) - \frac{1}{p^2}\frac{\partial}{\partial p}\left(\left[p^2 \dot{p}_{\text{loss}} + \frac{a_{zp} p^2}{L}\right] M\right)$$

$$= \begin{pmatrix} \partial_X \\ \partial_Y \\ \partial_z \\ p^{-2}\partial_p p^2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \kappa_{zz} & a_{zp} \\ 0 & 0 & -a_{zp} & A \end{pmatrix} \begin{pmatrix} \partial_X M \\ \partial_Y M \\ \partial_z M \\ \partial_p M \end{pmatrix} \tag{72}$$

with

$$a_{zp} = \frac{v a_p}{4} = \frac{v}{4}\int_{-1}^{1} d\mu \frac{(1-\mu^2) D_{\mu p}(\mu)}{D_{\mu\mu}(\mu)} = \frac{V_A H}{3} p, \tag{73}$$

for isospectral undamped slab Alfven wave turbulence, where $H = (I^+ - I^-)/(I^+ + I^-)$ denotes the net cross helicity of Alfven waves.

5.1. New transport terms due to weak adiabatic focusing

Adiabatic focusing gives rise to two terms in Eq. (72) that represent convective transport terms parallel to the guide field and in momentum space, respectively. In the limit $L \to \infty$ of negligible adiabatic focusing these terms vanish.

The convective term along the guide field has been derived before by Earl (1976) and Kunstmann (1979); the momentum convection term by RS and Shalchi (2008).

- For weak focusing ($|L| \gg \lambda$) the new parallel convective speed $\kappa_{zz}/L = v\lambda/3L$ is much less than the individual cosmic ray speed v.

- Particularly interesting is the new convection term in momentum space

$$\frac{1}{p^2}\frac{\partial}{\partial p}\left[\frac{a_{zp}p^2}{L}M\right] = \frac{1}{p^2}\frac{\partial}{\partial p}\left[\frac{V_A H}{3L}p^3 M\right]$$

  For positive values of the product $a_{zp}L \propto HL > 0$ it represents a continuous momentum loss term, whereas for negative values $a_{zp}L \propto HL < 0$ it represents a first-order Fermi-type acceleration term. The focusing length $L(z)$ is positive for a diverging guide magnetic field and negative for a converging guide field.

- This novel distributed focused acceleration process, which is a 1st order Fermi acceleration process, operates in all cosmic sources with $HL < 0$, including the upstream medium of shock waves, haloes of spiral galaxies and solar flare loops.

- For $HL > 0$ it represents a deceleration (momentum loss) process. It could prevent diffusive shock acceleration (or reduce the efficiency of the latter) in a diverging upstream magnetic field.

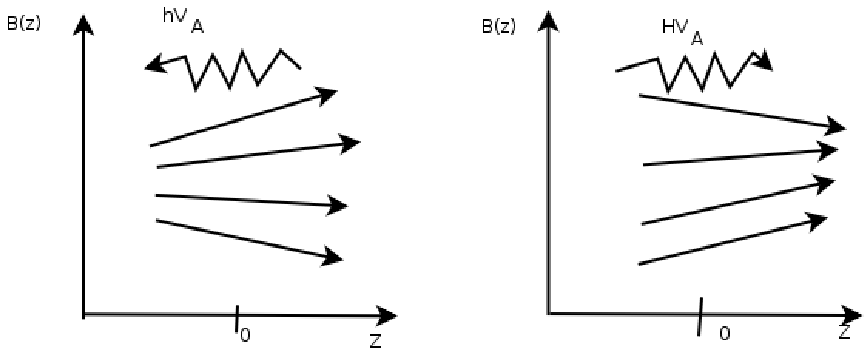

Figure 1: Conditions for 1st-order distributed Fermi acceleration in diverging (a) and converging (b) guide magnetic fields. In diverging magnetic fields a net negative ($h = -H < 0$) cross helicity state of Alfven waves (pronged curve) convects the average particle to regions of stronger field stength. In converging magnetic fields a net positive ($H > 0$) cross helicity state of Alfven waves also convects the average particle to regions of stronger field stength. In both cases the conservation of the pitch-angle averaged magnetic moment of the particle requires the increase of the particle momentum.

6. Summary and conclusions

- We have reviewed the derivation of the cosmic ray Fokker-Planck transport equation in turbulent nonuniform magnetic fields, explaining the physical assumptions underlying the derived Fokker-Planck coefficients.

- The non-uniformity of the large-scale guide magnetic field $\vec{B}_0$ provides via the mirror force additional adiabatic focusing of particles along the guide field characterized by the parallel focusing length L_3 (Roelof 1969, Earl 1974) and the gradient and curvature drifts of the cosmic ray guiding center perpendicular to the guide field characterized by the perpendicular field gradients L_1 and L_2.

- The cosmic ray diffusion approximation in the weak adiabatic focusing limit yields the modified diffusion-convection transport equation with many new diffusion and convection terms in the 4-dimensional momentum-position phase space (p, X, Y, z). The geometry and nature of the MHD fluctuations determine the individual transport terms. It is essential to know the cross helicity state of MHD turbulence in cosmic sources.

- For infinitely large perpendicular scale length ($L_1 = L_2 = \infty$), the focused diffusion-convection transport equation results exhibiting two convective terms parallel to the guide field and in momentum space. The respective convective speeds depend on the ratio of the parallel diffusion coefficient and the adiabatic deceleration rate to the focusing length, respüectively. For slab Alfvenic turbulence the adiabatic deceleration rate is proportional to the cross helicity $H \in [-1, 1]$ of the Alfven waves.

- For positive values of the product $HL > 0$ the new momentum convection term represents a continuous momentum loss term, whereas for negative values $HL < 0$ it represents a first-order Fermi-type acceleration term. L is positive for a diverging guide magnetic field and negative for a converging guide field.

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